

The Irrationality of $0.8 \cdot 0.98 \cdot 0.998 \cdot 0.9998 \dots$

Yoshihiro Paris
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109, U.S.
yparis@umich.edu

October 18, 2026

Abstract

In this paper, we prove the irrationality of a family of infinite products using a technique similar to that seen in Fourier’s proof of the irrationality of e . We also derive several consequences of the main result, the final one recovering Euler’s pentagonal number theorem through an algebraic telescoping argument which naturally demonstrates why the Pentagonal numbers emerge as consecutive sums in the exponent.

Introduction

What is the simplest way to prove the irrationality of the infinite product

$$0.8 \cdot 0.98 \cdot 0.998 \cdot 0.9998 \dots?$$

In [1], Fourier proved the irrationality of e . In this paper, we use a similar argument to provide an alternative proof of the irrationality of a range of numbers expressed as infinite products.

Theorem 1. *Let $A \in \mathbb{Z}$ with $|A| > 1$, and let $r = \frac{s}{t} \in \mathbb{Q}$ with $r \neq 0$, $r \neq -A^k$ for all $k \in \mathbb{Z}_{>0}$, and $s, t \in \mathbb{Z}$ with $\frac{s}{t}$ being in lowest terms. Suppose that $|s| < |A|$. Then*

$$D_A := \prod_{n=1}^{\infty} \left(1 + \frac{r}{A^n}\right), \tag{1}$$

is irrational.

Although the approach presented in this paper is technically involved, the proof is conceptually straightforward, with the nontechnical parts contained almost entirely on the page following Eq. (12). Fourier’s approach to proving the irrationality of e is well known and accessible, and this paper demonstrates that it can be adapted to a family of infinite products using only classical mathematical techniques.

Lotosky first proved a partial result for a related problem in 1943 [2]. The problem in full was originally proven by Bundschuh in 1969 [3], using properties of such infinite products as functional equations and analyzing their behavior. Here, we provide an alternative proof using a method analogous to Fourier’s proof of the irrationality of e . Lemmas 2 and 3 show, more generally, how infinite products of this form can be rewritten as infinite sums, and Lemma 4 gives a more compact expression in the case of infinite products of the form above. This is used to show that the denominators of the terms grow too quickly for the expression to take a rational value, with the core mechanism of the proof following directly from (12), as noted above.

The question posed in the title of this paper is answered in Example 6 using the same method as the main result, thereby compactly illustrating the argument in less than a page.

We additionally provide three propositions following from the main result. The last proposition recovers Euler’s pentagonal number theorem, whose best-known proof is the combinatorial proof by Franklin [4]. Although our approach is classical and shares similarity with Euler’s original proof [5], the distinctive feature of this presentation is that each telescoping step can effectively be viewed as exchanging a $q^n - 1$ factor from all denominators of an initial infinite sum representation for a power of q , from where the pentagonal numbers naturally accumulate. This is illustrated in Remark 11.

Outline of the Paper

Preliminary Results

Lemma 2. Let $n \geq 1$, and let $\{s_j\}_{1 \leq j \leq n} \subset \mathbb{R}$. The following equation holds:

$$\prod_{i=1}^n (1 + s_i) = 1 + \sum_{i=1}^n \sum_{1 \leq j_1 < \dots < j_i \leq n} s_{j_1} \cdots s_{j_i}. \quad (2)$$

Proof. We will proceed by induction. The case $n = 1$ is trivial.

(Inductive step) Assume that Eq. (2) holds for n , and consider $n + 1$. It follows that

$$\begin{aligned} \prod_{i=1}^{n+1} (1 + s_i) &= \left(1 + \sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_i \leq n} s_{j_1} \cdots s_{j_i} \right) \right) (1 + s_{n+1}) \\ &= 1 + \sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_i \leq n} s_{j_1} \cdots s_{j_i} \right) + s_{n+1} + s_{n+1} \sum_{i=1}^n \left(\sum_{1 \leq j_1 < \dots < j_i \leq n} s_{j_1} \cdots s_{j_i} \right) \\ &= 1 + \sum_{i=1}^{n+1} \left(\sum_{1 \leq j_1 < \dots < j_i \leq n+1} s_{j_1} \cdots s_{j_i} \right). \end{aligned}$$

As the base and the inductive step hold, Lemma 2 follows. $\square$

Lemma 3. Consider an infinite set $\{s_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$, and suppose that $\sum_{i=1}^{\infty} s_i$ is absolutely convergent. Then the following identity holds:

$$\prod_{i=1}^{\infty} (1 + s_i) = 1 + \sum_{i=1}^{\infty} \sum_{1 \leq j_1 < \dots < j_i} s_{j_1} \cdots s_{j_i}.$$

Proof. Since $\sum_{i=1}^{\infty} |s_i| < \infty$, we have:

$$\sum_{i=1}^{\infty} \log(1 + |s_i|) \leq \sum_{i=1}^{\infty} |s_i| =: C.$$

If $s_i = -1$ for any s_i , then the infinite product converges trivially. Else for all $n \in \mathbb{Z}_{>0}$,

$$\log \left| \prod_{i=1}^n (1 + s_i) \right| \leq \log \prod_{i=1}^n (1 + |s_i|) = \sum_{i=1}^n \log(1 + |s_i|) \leq C.$$

Thus, the infinite product converges absolutely. Consequently,

$$1 + \sum_{i=1}^{\infty} \sum_{1 \leq j_1 < \dots < j_i} |s_{j_1} \cdots s_{j_i}| = \prod_{i=1}^{\infty} (1 + |s_i|) < \infty.$$

Therefore, the elementary symmetric product is absolutely convergent, and taking the limit in Lemma 2 as $n \rightarrow \infty$ is justified. $\square$

Lemma 4. For $A, r \in \mathbb{R}$ where $|A| > 1$, the following holds:

$$D_A = \sum_{n=0}^{\infty} r^n \cdot \prod_{m=1}^n (A^m - 1)^{-1}. \quad (3)$$

Proof. It suffices to show that

$$\prod_{n=1}^{\infty} \left(1 + \frac{r}{A^n}\right) = \sum_{n=0}^{\infty} r^n \cdot \prod_{m=1}^n (A^m - 1)^{-1}.$$

First, observe

$$\sum_{n=1}^{\infty} \left| \frac{r}{A^n} \right| = \frac{|r|}{|A| - 1}$$

is absolutely convergent, hence Lemma 3 applies. Set $S = \{r \cdot A^{-j}\}_{j \in \mathbb{N}}$. By Lemma 3 we rewrite the right-hand side of Eq. (3) as:

$$\prod_{n=1}^{\infty} \left(1 + \frac{r}{A^n}\right) = 1 + \sum_{n=1}^{\infty} \left(r^n \sum_{1 \leq j_1 < \dots < j_n} A^{-j_1} \dots A^{-j_n} \right). \quad (4)$$

Therefore, it is enough to show that the following identity holds for all $n \in \mathbb{N}$.

$$\sum_{1 \leq j_1 < \dots < j_n} A^{-j_1} \dots A^{-j_n} = \prod_{m=1}^n (A^m - 1)^{-1}. \quad (5)$$

We will proceed by induction on n .

(Base step) For $n = 1$, Eq. (5) reduces to

$$\sum_{1 \leq j} A^{-j} = \frac{1}{A - 1} = \prod_{m=1}^1 (A^m - 1)^{-1}.$$

(Inductive step) assume that the formula holds for n ; that is,

$$\sum_{1 \leq j_1 < \dots < j_n} A^{-j_1} \dots A^{-j_n} = \prod_{m=1}^n (A^m - 1)^{-1}. \quad (6)$$

(Inductive case $n + 1$) For $k \geq 1$, we consider the term

$$\sum_{k=j_1 < \dots < j_{n+1}} A^{-j_1} \dots A^{-j_{n+1}}.$$

It follows that

$$\sum_{1 \leq j_1 < \dots < j_{n+1}} A^{-j_1} \dots A^{-j_{n+1}} = \sum_{k=1}^{\infty} \left(A^{-k} \cdot \left(\sum_{k+1 \leq j_2 < \dots < j_{n+1}} A^{-j_2} \dots A^{-j_{n+1}} \right) \right). \quad (7)$$

We factor out A^{-kn} , since each term in the nested sum in Eq. (7) contains the factor A^{-nk} . We then relabel the indices to obtain

$$\sum_{k=1}^{\infty} \left(A^{-(n+1)k} \left(\sum_{1 \leq j_1 < \dots < j_n} A^{-j_1} \dots A^{-j_n} \right) \right).$$

Substituting from Eq. (6), we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \left(A^{-(n+1)k} \cdot \prod_{m=1}^n (A^m - 1)^{-1} \right) &= \sum_{k=1}^{\infty} \left(A^{-(n+1)k} \right) \cdot \prod_{m=1}^n (A^m - 1)^{-1} \\ &= (A^{n+1} - 1)^{-1} \cdot \prod_{m=1}^n (A^m - 1)^{-1} = \prod_{m=1}^{n+1} (A^m - 1)^{-1}. \end{aligned}$$

Which proves the inductive step for Eq. (6).

Using Eq. (5), we rewrite Eq. (4) as follows:

$$\prod_{n=1}^{\infty} \left(1 + \frac{r}{A^n}\right) = \sum_{n=0}^{\infty} r^n \cdot \prod_{m=1}^n (A^m - 1)^{-1},$$

where we use the convention that $\prod_{m=1}^0 (A^m - 1)^{-1} = 1$ □

Main Proof

Proof of Theorem 1. Let D_A be given by Eq. (1) for some $A \in \mathbb{Z}$ where $|A| > 1$, and $r = \frac{s}{t} \in \mathbb{Q} \setminus \{0\}$. Let us assume that $|s| < |A|$ and $-A^k \neq \frac{s}{t}$ for $s, t \in \mathbb{Z}$ in lowest terms and $k \in \mathbb{Z}_{>0}$. Given any fixed positive integer a , there exists a positive integer n' , such that for all $n \geq n'$ the following identities hold:

$$\left| \frac{A^n - 1}{s^n} \right| > a + 1 \quad \text{and} \quad |A^n - 1| > a + 1. \quad (8)$$

Indeed, both $\left| \frac{A^n - 1}{s^n} \right|$ and $|A^n - 1|$ strictly increase without bound. We first establish the following estimate:

$$\begin{aligned} \left| \sum_{n=n'}^{\infty} \frac{s^n}{t^{n-n'}} \cdot \prod_{m=1}^n (A^m - 1)^{-1} \right| &\leq \sum_{n=n'}^{\infty} \left| \frac{s^n}{t^{n-n'}} \cdot \prod_{m=1}^n (A^m - 1)^{-1} \right| \\ &= \left| \prod_{m=1}^{n'-1} (A^m - 1)^{-1} \right| \cdot \sum_{n=n'}^{\infty} \left| \frac{s^n}{t^{n-n'}} \cdot \prod_{m=n'}^n (A^m - 1)^{-1} \right| \\ &= \left| \prod_{m=1}^{n'-1} (A^m - 1)^{-1} \right| \cdot \sum_{n=n'}^{\infty} \left| \left(t^{n-n'} \cdot \frac{A^n - 1}{s^n} \right)^{-1} \cdot \prod_{m=n'}^{n-1} (A^m - 1)^{-1} \right|. \end{aligned}$$

For $n \geq n'$, Eq. (8) gives $\left| t^{n-n'} \frac{A^n - 1}{s^n} \right| > a + 1$. Combining this with the second inequality in Eq. (8), we obtain

$$\begin{aligned} \left| \sum_{n=n'}^{\infty} \frac{s^n}{t^{n-n'}} \cdot \prod_{m=1}^n (A^m - 1)^{-1} \right| &< \left| \prod_{m=1}^{n'-1} (A^m - 1)^{-1} \right| \cdot \sum_{n=n'}^{\infty} \left| (a+1)^{-1} \cdot \prod_{m=n'}^{n-1} (a+1)^{-1} \right| \\ &= \left| \prod_{m=1}^{n'-1} (A^m - 1)^{-1} \right| \cdot \sum_{n=1}^{\infty} \left| \prod_{m=1}^n (a+1)^{-1} \right| \\ &= \left| \prod_{m=1}^{n'-1} (A^m - 1)^{-1} \right| \cdot \sum_{n=1}^{\infty} (a+1)^{-n} = \left| \frac{1}{a} \cdot \prod_{m=1}^{n'-1} (A^m - 1)^{-1} \right|. \end{aligned} \quad (9)$$

Similarly, we establish a second estimate:

$$\begin{aligned} \left| \sum_{n=n'}^{\infty} r^n \prod_{m=1}^n (A^m - 1)^{-1} \right| &\geq \left| r^{n'} \prod_{m=1}^{n'} (A^m - 1)^{-1} \right| - \sum_{n=n'}^{\infty} \left| r^{n+1} \prod_{m=1}^{n+1} (A^m - 1)^{-1} \right| \\ &= \left| r^{n'} \prod_{m=1}^{n'} (A^m - 1)^{-1} \right| - \left| r^{n'} \prod_{m=1}^{n'} (A^m - 1)^{-1} \right| \cdot \sum_{n=n'}^{\infty} \left| r^{n-n'+1} \prod_{m=n'}^{n+1} (A^m - 1)^{-1} \right| \\ &= \left| r^{n'} \prod_{m=1}^{n'} (A^m - 1)^{-1} \right| - \left| r^{n'} \prod_{m=1}^{n'} (A^m - 1)^{-1} \right| \cdot \sum_{n=n'}^{\infty} \left| \left(\frac{A^{n+1} - 1}{r^{n-n'+1}} \right)^{-1} \prod_{m=n'}^n (A^m - 1)^{-1} \right|. \end{aligned} \quad (10)$$

Using $n \geq n'$ and as $\left| \frac{A^{n+1}-1}{r^{n-n'+1}} \right| = \left| t^{n-n'+1} \frac{A^{n+1}-1}{s^{n-n'+1}} \right| > \left| t^{n-n'} \frac{A^n-1}{s^n} \right| > a+1$, and also using the second inequality from Eq. (8), it follows that

$$\begin{aligned}
\left| \sum_{n=n'}^{\infty} r^n \prod_{m=1}^n (A^m-1)^{-1} \right| &> \left| r^{n'} \prod_{m=1}^{n'} (A^m-1)^{-1} \right| - \left| r^{n'} \prod_{m=1}^{n'} (A^m-1)^{-1} \right| \cdot \sum_{n=n'}^{\infty} \left| (a+1)^{-1} \prod_{m=n'}^n (a+1)^{-1} \right| \\
&= \left| r^{n'} \prod_{m=1}^{n'} (A^m-1)^{-1} \right| - \left| r^{n'} \prod_{m=1}^{n'} (A^m-1)^{-1} \right| \cdot \sum_{n=1}^{\infty} (a+1)^{-n} \\
&\geq \left| r^{n'} \prod_{m=1}^{n'} (A^m-1)^{-1} \right| - \left| r^{n'} \prod_{m=1}^{n'} (A^m-1)^{-1} \right| \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = 0.
\end{aligned} \tag{11}$$

We express D_A in the form:

$$D_A = \sum_{n=0}^{\infty} r^n \cdot \prod_{m=1}^n (A^m-1)^{-1} = P + C, \tag{12}$$

where

$$P := \sum_{n=0}^{n'-1} r^n \cdot \prod_{m=1}^n (A^m-1)^{-1}, \text{ and } C := \sum_{n=n'}^{\infty} r^n \cdot \prod_{m=1}^n (A^m-1)^{-1}.$$

We will proceed by contradiction. Assume that D_A is a rational number. Hence $a \cdot D_A = b$ for some positive integer a and some integer b . Note that C must be non-zero as shown in Eq. (11). Then, we proceed with the following simplifications:

$$\begin{aligned}
a \cdot (P + C) &= b, \\
a \cdot t^{n'} \cdot \prod_{m=1}^{n'-1} (A^m-1) \cdot (P + C) &= b \cdot t^{n'} \cdot \prod_{m=1}^{n'-1} (A^m-1) \\
a \cdot \sum_{n=0}^{n'-1} \left(s^n t^{n'-n} \cdot \prod_{m=n+1}^{n'-1} (A^m-1) \right) + a \cdot t^{n'} \cdot C \cdot \prod_{m=1}^{n'-1} (A^m-1) &= b \cdot t^{n'} \cdot \prod_{m=1}^{n'-1} (A^m-1).
\end{aligned}$$

Here,

$$z_1 := a \cdot \sum_{n=0}^{n'-1} \left(s^n t^{n'-n} \cdot \prod_{m=n+1}^{n'-1} (A^m-1) \right), \text{ and } z_2 := b \cdot t^{n'} \cdot \prod_{m=1}^{n'-1} (A^m-1).$$

are integers, since they are finite sums and products of integers. Thus,

$$a \cdot t^{n'} \cdot C \cdot \prod_{m=1}^{n'-1} (A^m-1) = z_2 - z_1. \tag{13}$$

The right-hand side is an integer, since the integers are closed under subtraction. By definition of C :

$$t^{n'} \cdot C = \sum_{n=n'}^{\infty} \frac{s^n}{t^{n-n'}} \cdot \prod_{m=1}^n (A^m-1)^{-1}.$$

Since $t \neq 0$, Eqs. (11) and (9) imply that

$$0 < \left| t^{n'} \cdot C \right| < \left| \frac{1}{a} \cdot \prod_{m=1}^{n'-1} (A^m-1)^{-1} \right|.$$

Thus,

$$0 < \left| a \cdot t^{n'} \cdot C \cdot \prod_{m=1}^{n'-1} (A^m-1) \right| < \left| \prod_{m=1}^{n'-1} (A^m-1) \cdot a \cdot \frac{1}{a} \cdot \prod_{m=1}^{n'-1} (A^m-1)^{-1} \right| = 1.$$

This is a contradiction, since it implies that the left-hand side of Eq. (13) is a nonzero integer with magnitude less than 1. Hence, D_A cannot be rational. $\square$

The following example illustrates the simplest case, where the similarity to Fourier's proof is very visible.

Example 5. *The infinite product*

$$1.1 \cdot 1.01 \cdot 1.001 \cdots$$

is irrational.

Proof. We will use the following equivalence: $r \in \mathbb{R}$ is irrational if and only if $ar \notin \mathbb{Z}$ for all $a \in \mathbb{N} \setminus \{0\}$.

First, Lemma 4 gives the identity

$$\sum_{n=0}^{\infty} \prod_{m=1}^n (10^m - 1)^{-1} = 1 + \frac{1}{9} + \frac{1}{9 \cdot 99} + \frac{1}{9 \cdot 99 \cdot 999} + \cdots.$$

Choose any positive integer a , and let k be a sufficiently large positive integer such that $a < 10^k - 1$. Multiplying the preceding identity by $a \prod_{i=1}^k (10^i - 1)$, we obtain

$$\begin{aligned} a \prod_{i=1}^k (10^i - 1) \left(1 + \frac{1}{9} + \frac{1}{9 \cdot 99} + \cdots \right) &= a \prod_{i=1}^k (10^i - 1) \left(1 + \frac{1}{9} + \cdots + \frac{1}{9 \cdot 99 \cdots (10^k - 1)} \right) \\ &\quad + a \left(\frac{1}{10^{k+1} - 1} + \frac{1}{(10^{k+1} - 1)(10^{k+2} - 1)} + \cdots \right). \end{aligned}$$

The first term on the right-hand side of the preceding equation is an integer. It remains to show that the second term lies strictly between 0 and 1. Using a geometric series estimate, for every such k , we have

$$\begin{aligned} 0 &< \frac{1}{10^{k+1} - 1} + \frac{1}{(10^{k+1} - 1)(10^{k+2} - 1)} + \cdots \\ &< \sum_{i=1}^{\infty} \frac{1}{(10^{k+1} - 1)^i} = \frac{1}{10^{k+1} - 2} < \frac{1}{10^k - 1} < \frac{1}{a}. \end{aligned}$$

Therefore,

$$0 < a \left(\frac{1}{10^{k+1} - 1} + \frac{1}{(10^{k+1} - 1)(10^{k+2} - 1)} + \cdots \right) < 1.$$

Thus, for all positive integers a , the number $a(1.1 \cdot 1.01 \cdot 1.001 \cdots) \notin \mathbb{Z}$. By the equivalence mentioned above, this proves that $1.1 \cdot 1.01 \cdot 1.001 \cdots$ is irrational. $\square$

The following example illustrates the paper's title.

Example 6. *The infinite product*

$$0.8 \cdot 0.98 \cdot 0.998 \cdots$$

is irrational.

Proof. We use the same strategy as in Example 5. First, Lemma 4 gives the identity

$$\sum_{n=0}^{\infty} (-2)^n \prod_{m=1}^n (10^m - 1)^{-1} = 1 - \frac{2}{9} + \frac{2^2}{9 \cdot 99} - \frac{2^3}{9 \cdot 99 \cdot 999} + \cdots.$$

Choose any positive integer a , and let k be a sufficiently large positive integer such that $a < (10^k - 1)/2^k$. Define

$$B_k := \frac{(-2)^{k+2}}{(10^{k+1} - 1)(10^{k+2} - 1)} + \frac{(-2)^{k+3}}{(10^{k+1} - 1)(10^{k+2} - 1)(10^{k+3} - 1)} + \cdots.$$

Multiplying the preceding identity by $a \prod_{i=1}^k (10^i - 1)$, we obtain

$$a \prod_{i=1}^k (10^i - 1) \left(1 - \frac{2}{9} + \frac{2^2}{9 \cdot 99} - \cdots \right) = a \prod_{i=1}^k (10^i - 1) \left(1 - \frac{2}{9} + \frac{2^2}{9 \cdot 99} - \cdots + \frac{(-2)^k}{9 \cdot 99 \cdots (10^k - 1)} \right) + a \left(\frac{(-2)^{k+1}}{10^{k+1} - 1} + B_k \right).$$

The first term on the right-hand side of the preceding equation is an integer. It remains to show that the second term has absolute value strictly between 0 and 1. Using the triangle inequality and a geometric series estimate, for every such k , we obtain

$$\begin{aligned} |B_k| &= \left| \frac{(-2)^{k+2}}{(10^{k+1} - 1)(10^{k+2} - 1)} + \frac{(-2)^{k+3}}{(10^{k+1} - 1)(10^{k+2} - 1)(10^{k+3} - 1)} \cdots \right| \\ &< \frac{2^{k+2}}{(10^{k+1} - 1)(10^{k+2} - 1)} + \frac{2^{k+3}}{(10^{k+1} - 1)(10^{k+2} - 1)(10^{k+3} - 1)} \cdots \\ &< \frac{2^{k+1}}{10^{k+1} - 1} \sum_{i=1}^{\infty} \left(\frac{2}{10^{k+1} - 1} \right)^i \\ &= \frac{2^{k+1}}{10^{k+1} - 1} \frac{2}{10^{k+1} - 3} < \frac{2^{k+1}}{10^{k+1} - 1}. \end{aligned}$$

From this, it follows that

$$0 < \left| \frac{(-2)^{k+1}}{10^{k+1} - 1} + B_k \right|.$$

Using a similar estimate, we obtain

$$\begin{aligned} \left| \frac{(-2)^{k+1}}{10^{k+1} - 1} + B_k \right| &= \left| \frac{(-2)^{k+1}}{(10^{k+1} - 1)} + \frac{(-2)^{k+2}}{(10^{k+1} - 1)(10^{k+2} - 1)} + \cdots \right| \\ &< \frac{2^{k+1}}{(10^{k+1} - 1)} + \frac{2^{k+2}}{(10^{k+1} - 1)(10^{k+2} - 1)} + \cdots \\ &< 2^k \sum_{i=1}^{\infty} \left(\frac{2}{10^{k+1} - 1} \right)^i \\ &< \frac{2^k}{10^k - 1} < \frac{1}{a}. \end{aligned}$$

Combining these two estimates, we conclude that

$$0 < \left| a \left(\frac{(-2)^{k+1}}{10^{k+1} - 1} + B_k \right) \right| < 1.$$

Thus, for every positive integer a , the number $a(0.8 \cdot 0.98 \cdot 0.998 \cdots) \notin \mathbb{Z}$. Hence, $0.8 \cdot 0.98 \cdot 0.998 \cdots$ is irrational. $\square$

Consequences

Here, we present three related identities.

First Consequence

In this proposition, we derive a general expression for the infinite product

$$\prod_{i=1}^{\infty} (1 + r/A^i)^n$$

as an infinite sum, where $r, A \in \mathbb{R} \setminus \{0\}$, $|A| > 1$, and $n \in \mathbb{N}$. Since this is effectively a power-series expansion of an infinite product, one might expect the resulting expression to be unwieldy. We include the result because the solution simplifies surprisingly well through combinatorial arguments.

Proposition 7. Fix $n \in \mathbb{N}$. Let $A \in \mathbb{R}$ with $|A| > 1$. Let $a_i : \mathbb{Z} \rightarrow \mathbb{R}$ be the sequence defined by the recurrence

$$a_i = \sum_{k=1}^n \binom{n}{k} \frac{a_{i-k}}{A^i - 1}, \quad a_0 = 1 \text{ and } a_i = 0 \text{ for all } i < 0.$$

Then the power series

$$F(r) = \sum_{i=0}^{\infty} a_i r^i$$

is absolutely convergent and has infinite radius of convergence. Furthermore, the following identity holds:

$$F(r) = \prod_{i=1}^{\infty} \left(1 + \frac{r}{A^i}\right)^n. \quad (14)$$

Proof. We first show that $F(r)$ is absolutely convergent and has an infinite radius of convergence. Let $R = |r| > 0$, and define $u_i := |a_i| R^i$, where $u_i = 0$ for $i < 0$. By the recurrence defining a_i , we obtain

$$u_i \leq \frac{1}{|A^i - 1|} \sum_{k=1}^n \binom{n}{k} R^k u_{i-k}.$$

Define

$$C := (R + 1)^n - 1.$$

Since $R > 0$, we have

$$\sum_{k=1}^n \binom{n}{k} R^k u_{i-k} \leq \left(\sum_{k'=1}^n \binom{n}{k'} R^{k'} \right) \left(\sum_{k=1}^n u_{i-k} \right) = C \sum_{k=1}^n u_{i-k}.$$

There exists integer N such that

$$\frac{C}{|A^i - 1|} < \frac{1}{2n}$$

for all $i \geq N$, since C is independent of i . Define

$$S_M := \sum_{i=N}^M u_i.$$

Then

$$S_M \leq \frac{1}{2n} \sum_{i=N}^M \sum_{k=1}^n u_{i-k}.$$

Each u_i with $N \leq i \leq M$ appears at most n times, so that part of the sum is bounded above by $\frac{1}{2} \sum_{i=N}^M u_i = \frac{1}{2} S_M$. For terms u_i with index below N , define

$$C' := \frac{1}{2} \sum_{i=N-n}^{N-1} u_i.$$

The contribution from indices less than N is independent of M , so C' is constant with respect to M . Thus,

$$S_M \leq \frac{1}{2} S_M + C',$$

and hence

$$S_M \leq 2C'.$$

Since this inequality holds for all M , and since $\sum_{i=0}^N u_i$ is finite,

$$|F(r)| \leq \sum_{i=0}^{\infty} |a_i| r^i = \sum_{i=0}^{\infty} u_i$$

converges absolutely. Since R was arbitrary, the radius of convergence is infinite.

We proceed to prove the identity from Eq. (14). By the definition of a_i , we obtain:

$$a_i = \sum_{k=1}^n \binom{n}{k} \frac{a_{i-k}}{A^i - 1} = \sum_{k=1}^n \binom{n}{k} \frac{A^{-i}}{1 - A^{-i}} a_{i-k}.$$

It follows that

$$a_i - A^{-i} a_i = A^{-i} \sum_{k=1}^n \binom{n}{k} a_{i-k},$$

and hence

$$a_i = A^{-i} a_i + A^{-i} \sum_{k=1}^n \binom{n}{k} a_{i-k} = A^{-i} \sum_{k=0}^n \binom{n}{k} a_{i-k}.$$

Note that

$$F\left(\frac{r}{A}\right) = \sum_{i=0}^{\infty} a_i \left(\frac{r}{A}\right)^i,$$

and

$$\sum_{i=0}^{\infty} a_{i-k} \left(\frac{r}{A}\right)^{i-k} = \sum_{i=0}^{\infty} a_i \left(\frac{r}{A}\right)^i = F\left(\frac{r}{A}\right),$$

since $a_{i-k} = 0$ whenever $k > i$. Thus,

$$\begin{aligned} F(r) &= \sum_{i=0}^{\infty} r^i A^{-i} \sum_{k=0}^n \binom{n}{k} a_{i-k} = F\left(\frac{r}{A}\right) + \sum_{i=0}^{\infty} r^i A^{-i} \sum_{k=1}^n \binom{n}{k} a_{i-k} \\ &= F\left(\frac{r}{A}\right) + \sum_{k=1}^n \binom{n}{k} \left(\frac{r}{A}\right)^k \sum_{i=0}^{\infty} a_{i-k} \left(\frac{r}{A}\right)^{i-k} \\ &= F\left(\frac{r}{A}\right) + F\left(\frac{r}{A}\right) \sum_{k=1}^n \binom{n}{k} \left(\frac{r}{A}\right)^k \\ &= F\left(\frac{r}{A}\right) \sum_{k=0}^n \binom{n}{k} \left(\frac{r}{A}\right)^k \\ &= F\left(\frac{r}{A}\right) \left(1 + \frac{r}{A}\right)^n. \end{aligned}$$

Therefore,

$$F\left(\frac{r}{A^i}\right) = \left(1 + \frac{r}{A^i}\right)^n F\left(\frac{r}{A^{i+1}}\right).$$

Iterating this identity gives

$$\sum_{i=0}^{\infty} a_i r^i = F(r) = \prod_{i=1}^{\infty} \left(1 + \frac{r}{A^i}\right)^n.$$

□

Second Consequence

Consider $f : \mathbb{N} \rightarrow \mathbb{R}$ such that $\sum_{n \in \mathbb{N}} f(n)$ is absolutely convergent. In this proposition, we express the infinite product

$$\prod_{n=1}^{\infty} (1 + f(n))$$

as a series whose terms are determined recursively by the power sums of $f(n)$. This recurrence is an infinite-variable analog of the Newton-Girard identities relating elementary symmetric functions and power sums [6, 7]. Since the expression simplifies inductively to a relatively compact recurrence, we include the proof.

Proposition 8. *Let $f : \mathbb{N} \rightarrow \mathbb{R}$. If*

$$\sum_{n=1}^{\infty} f(n)$$

converges absolutely, then the following identity holds:

$$\prod_{n=1}^{\infty} (1 + f(n)) = \sum_{n=0}^{\infty} A_n,$$

where $A_0 = 1$, and for all $k \geq 1$,

$$A_k := -\frac{1}{k} \sum_{i=0}^{k-1} A_i \sum_{j=1}^{\infty} (-f(j))^{k-i}. \quad (15)$$

Proof. By Lemma 3, we rewrite the product as

$$\prod_{n=1}^{\infty} (1 + f(n)) = 1 + \sum_{n=1}^{\infty} \left(\sum_{1 \leq j_1 < \dots < j_n} f(j_1) \cdots f(j_n) \right). \quad (16)$$

It remains to identify the parenthesized expression in (16) with A_k . To this end, define

$$\tilde{A}_k := \sum_{1 \leq j_1 < \dots < j_k} f(j_1) \cdots f(j_k). \quad (17)$$

First, we verify the base case $A_1 = \tilde{A}_1$. Indeed, by the definition of $\tilde{A}_1$, we obtain

$$\tilde{A}_1 = \sum_{1 \leq j_1} f(j_1) = \sum_{j=1}^{\infty} f(j) = \frac{1}{1} \sum_{i=0}^0 A_0 \sum_{j=1}^{\infty} (f(j))^{1-i} = A_1.$$

Second, we present the following relation:

$$\sum_{\substack{1 \leq j \\ j \neq i}} f(j) = \left(\sum_{1 \leq j} f(j) \right) - f(i) = A_1 - f(i).$$

We will proceed by induction. Let all $k' < k$ satisfy $\tilde{A}_{k'} = A_{k'}$, and let $k \geq 2$. It follows:

$$\tilde{A}_k := \sum_{1 \leq j_1 < \dots < j_k} f(j_1) \cdots f(j_k) = \sum_{1 \leq j_1} f(j_1) \sum_{j_1 < j_2 < \dots < j_k} f(j_2) \cdots f(j_k).$$

By relabeling j_1 by i , and shifting all indices of j down by 1, we obtain

$$\begin{aligned} \tilde{A}_k &= \sum_{i=1}^{\infty} f(i) \sum_{i < j_1 < \dots < j_{k-1}} f(j_1) \cdots f(j_{k-1}) \\ &= \frac{1}{k} \sum_{i=1}^{\infty} f(i) \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \\ j_1, \dots, j_{k-1} \neq i}} f(j_1) \cdots f(j_{k-1}) \end{aligned} \quad (18)$$

In the above equation, we used the following argument: when we restrict the indices by the condition $\ell < j_1 < \dots < j_{k-1}$, the term $f(\ell)f(j_1)f(j_2)\cdots f(j_{k-1})$ appears exactly once in the first sum for each $\ell \in \mathbb{N}$. But when we allow the indices $j_1, \dots, j_{k-1}$ to be smaller than i , the term $f(\ell)f(j_1)f(j_2)\cdots f(j_{k-1})$ appear k times in the second sum. For instance, the term appears when $i = \ell$ and $\ell < j_1 < \dots < j_{k-1}$, or when $i = j_1$ and $j_1 < \ell < \dots < j_{k-1}$, and similarly for the remaining choices of i . To summarize, in the first sum, we add over the ordered set of k indices. In the second sum, we add over the same set, but in an unordered way, once for each possible choice of the distinguished index i . In that case, each product is counted exactly k times.

From Eq. (18), and by an index counting argument, we obtain that

$$\sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \\ j_1, \dots, j_{k-1} \neq i}} f(j_1) \cdots f(j_{k-1}) = \sum_{1 \leq j_1 < \dots < j_{k-1}} \left(f(j_1) \cdots f(j_{k-1}) \right) - f(i) \sum_{\substack{1 \leq j'_1 < \dots < j'_{k-2} \\ j'_1, \dots, j'_{k-2} \neq i}} \left(f(j'_1) \cdots f(j'_{k-2}) \right)$$

Using the above equation, Eq. (17) and the inductive hypothesis $A_i = \tilde{A}_i$ for $i < k$, we obtain

$$\begin{aligned} \tilde{A}_k &= \frac{1}{k} \sum_{i=1}^{\infty} f(i) \left(\sum_{1 \leq j_1 < \dots < j_{k-1}} \left(f(j_1) \cdots f(j_{k-1}) \right) - f(i) \sum_{\substack{1 \leq j'_1 < \dots < j'_{k-2} \\ j'_1, \dots, j'_{k-2} \neq i}} \left(f(j'_1) \cdots f(j'_{k-2}) \right) \right) \\ &= \frac{1}{k} \sum_{i=1}^{\infty} f(i) \left(A_{k-1} - f(i) \sum_{\substack{1 \leq j_1 < \dots < j_{k-2} \\ j_1, \dots, j_{k-2} \neq i}} \left(f(j_1) \cdots f(j_{k-2}) \right) \right). \end{aligned}$$

We see that the above equation contains the exact same sum as seen in Eq. (18), except the index is one short. Thus, the argument is repeated inductively, from where it follows that

$$\begin{aligned} \tilde{A}_k &= \frac{1}{k} \sum_{i=1}^{\infty} f(i) \left(A_{k-1} - f(i) \left(A_{k-2} - f(i) \sum_{\substack{1 \leq j_1 < \dots < j_{k-3} \\ j_1, \dots, j_{k-3} \neq i}} \left(f(j_1) \cdots f(j_{k-3}) \right) \right) \right) \\ &= -\frac{1}{k} \sum_{i=1}^{\infty} \left(-f(i) A_{k-1} + A_{k-2} (-f(i))^2 + (-f(i))^3 \sum_{\substack{1 \leq j_1 < \dots < j_{k-3} \\ j_1, \dots, j_{k-3} \neq i}} \left(f(j_1) \cdots f(j_{k-3}) \right) \right) \end{aligned}$$

By repeating this process $k-1$ times, we conclude the following:

$$\tilde{A}_k = -\frac{1}{k} \left(\left(\sum_{i=1}^{k-1} A_i \sum_{j=1}^{\infty} (-f(j))^{k-i} \right) + \sum_{i=1}^{\infty} (-f(i))^k \right).$$

Finally, as $A_0 = 1$, we write:

$$\tilde{A}_k = -\frac{1}{k} \sum_{i=0}^{k-1} A_i \sum_{j=1}^{\infty} (-f(j))^{k-i}.$$

The above expression is A_k by the definition given in Eq. (15). Therefore the relation $A_k = \tilde{A}_k$ holds for $k \geq 1$, and Proposition 8 follows. $\square$

Third Consequence

The goal of this section is to rederive Euler's pentagonal number theorem in a natural manner from Lemma 4.

In the following proposition, we will introduce the notation $a_k! := \prod_{n=1}^k a_n$ for some given sequence a_k , where we define the empty product $a_0! := 1$. Specifically, we will consider the sequence $a_n := (q^n - 1)$. Let us also introduce a shorthand notation for the truncated version with a shifted lower bound:

$${}_j a_k! := \frac{a_k!}{a_{j-1}!} = \prod_{n=j}^k a_n$$

for integers $1 \leq j \leq k$, and where ${}_j a_k! = 1$ for $k < j$.

Note that $q_k!$ may be expressed by the standard q-Pochhammer notation $(-1)^k(q; q)_k$ and ${}_j q_k!$ as $(-1)^{k-j+1}(q^j; q)_{k-j+1}$ [8]. We will use the notation introduced above for compactness and clarity in the following proof.

Proposition 9. *Lemma 4 recovers Euler's pentagonal number theorem for the special case $r = -1$. More precisely, if $q \in \mathbb{R}$ is such that $1 < |q|$ then*

$$D_q = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{(3k^2-k)/2}}.$$

Remark 10. Here we will use the notation D_q rather than D_A as in Eq. (3) since this is the standard notation of Euler's pentagonal number theorem. In the following proof, the notation gets rather dense because of the formalization. A simplified illustration of the mechanism of the proof is provided in Remark 11.

Proof. Consider D_q from Eq. (3) for $q \in \mathbb{R}, 1 < |q|$ and for fixed $r = -1$. It follows from Lemma 4:

$$D_q = \sum_{k=0}^{\infty} \left((-1)^k \cdot \prod_{n=1}^k (q^n - 1)^{-1} \right).$$

Consider the function $T : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $T(a) := \frac{a^2 - a}{2}$. It follows from the definition that

$$T(a) + a = \frac{a^2 + a}{2} = \frac{(a+1)^2 - (a+1)}{2} = T(a+1).$$

Using the notations introduced above, we see that D_q has the form:

$$D_q = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{q_k!}.$$

We now write D_q as a finite sum plus a remainder. The remainder will be decomposed into the next two pentagonal terms and a new remainder. For $a, k \in \mathbb{N}$, define

$${}_a C_k := \frac{(-1)^k}{{}_a q_k! \cdot q^{(a-1)k+T(a)}}.$$

Thus for $a = 1$, this gives

$$D_q = 1 + \sum_{k=1}^{\infty} {}_1 C_k. \tag{19}$$

Now fix an arbitrary positive integer. We decompose ${}_a C_k$ as follows:

$${}_a C_k = \frac{(-1)^k}{{}_a q_k! \cdot q^{(a-1)k+T(a)}} = \frac{(-1)^k}{{}_a q_{k-1}! \cdot q^{ak+T(a)}} + \frac{(-1)^k}{{}_a q_k! \cdot q^{ak+T(a)}}.$$

If we define ${}_a A_k$ and ${}_a B_k$ as follows:

$${}_a A_k := \frac{(-1)^k}{{}_a q_{k-1}! \cdot q^{ak+T(a)}}, \text{ and } {}_a B_k := \frac{(-1)^k}{{}_a q_k! \cdot q^{ak+T(a)}},$$

then the previous decomposition becomes

$${}_a C_k = {}_a A_k + {}_a B_k.$$

Observe that

$$\begin{aligned} {}_a B_k + {}_a A_{k+1} &= \frac{(-1)^k}{{}_a q_k! \cdot q^{ak+T(a)}} - \frac{(-1)^k}{{}_a q_k! \cdot q^{a(k+1)+T(a)}} \\ &= (-1)^k \frac{q^a - 1}{{}_a q_k! \cdot q^{ak+a+T(a)}} \\ &= \frac{(-1)^k}{{}_{a+1} q_k! \cdot q^{ak+T(a+1)}} = {}_{a+1} C_k. \end{aligned}$$

Using this relation, we rewrite the tail sum as follows:

$$\begin{aligned}
\sum_{k=a}^{\infty} {}_a C_k &= \sum_{k=a}^{\infty} ({}_a A_k + {}_a B_k) \\
&= {}_a A_a + \sum_{k=a}^{\infty} ({}_a B_k + {}_a A_{k+1}) \\
&= {}_a A_a + \sum_{k=a}^{\infty} {}_{a+1} C_k \\
&= {}_a A_a + {}_{a+1} C_a + \sum_{k=a+1}^{\infty} {}_{a+1} C_k.
\end{aligned}$$

When $k = a$, the product ${}_a q_{(a-1)}!$ is empty, so it equals 1. Hence

$${}_a A_a = \frac{(-1)^a}{{}_a q_{(a-1)}! \cdot q^{aa+T(a)}} = \frac{(-1)^a}{q^{a^2+T(a)}},$$

Similarly, the product ${}_{a+1} q_a!$ is empty, so

$${}_{a+1} C_a = \frac{(-1)^a}{q^{a^2+T(a+1)}}.$$

Therefore,

$$\sum_{k=a}^{\infty} {}_a C_k = \frac{(-1)^a}{q^{a^2+T(a)}} + \frac{(-1)^a}{q^{a^2+T(a+1)}} + \sum_{k=a+1}^{\infty} {}_{a+1} C_k.$$

Since the relation holds for all a , we substitute it into Eq. (19):

$$D_q = 1 + \sum_{k=1}^{\infty} {}_1 C_k = 1 + \sum_{k=1}^{a-1} \left(\frac{(-1)^k}{q^{k^2+T(k)}} + \frac{(-1)^k}{q^{k^2+T(k+1)}} \right) + \sum_{k=a}^{\infty} {}_a C_k.$$

Remark 11. The notation above is mainly bookkeeping for a relatively simple rewriting. The first few rewritings (indexed by a in the above notation) follow this schematic form:

$$\begin{aligned}
D_q &= 1 - \frac{1}{(q-1)} + \frac{1}{(q-1)(q^2-1)} - \frac{1}{(q-1)(q^2-1)(q^3-1)} + \frac{1}{(q-1)(q^2-1)(q^3-1)(q^4-1)} - \cdots \\
&= 1 - \frac{1}{q} - \frac{1}{q^2} + \frac{1}{q^3} \frac{1}{(q^2-1)} - \frac{1}{q^4} \frac{1}{(q^2-1)(q^3-1)} + \frac{1}{q^5} \frac{1}{(q^2-1)(q^3-1)(q^4-1)} - \cdots \\
&= 1 - \frac{1}{q} - \frac{1}{q^2} + \frac{1}{q^{3+2}} + \frac{1}{q^{4+3}} - \frac{1}{q^{5+4}} \frac{1}{(q^3-1)} + \frac{1}{q^{6+5}} \frac{1}{(q^3-1)(q^4-1)} - \cdots \\
&= 1 - \frac{1}{q} - \frac{1}{q^2} + \frac{1}{q^{3+2}} + \frac{1}{q^{4+3}} - \frac{1}{q^{5+4+3}} - \frac{1}{q^{6+5+4}} + \frac{1}{q^{7+6+5}} \frac{1}{(q^4-1)} - \cdots \\
&\Rightarrow 1 - \frac{1}{q} - \frac{1}{q^2} + \frac{1}{q^5} + \frac{1}{q^7} - \frac{1}{q^{12}} - \frac{1}{q^{15}} + \frac{1}{q^{22}} + \frac{1}{q^{26}} - \cdots.
\end{aligned}$$

We see the pentagonal numbers naturally emerge as consecutive sums:

$$1, 2; \quad 3+2, 4+3; \quad 5+4+3, 6+5+4; \quad \cdots$$

When split into $1, 5, 12, 22, \dots$ and $2, 7, 15, 26, \dots$ we get the terms in the sequence $(3k^2 - k)/2$ for positive and negative indexes respectively. We will proceed by formally proving that the remainder converges to 0 as a approaches ∞ .

Consider

$$\lim_{a \rightarrow \infty} \sum_{k=a}^{\infty} {}_a C_k = \lim_{a \rightarrow \infty} \sum_{k=a}^{\infty} \frac{(-1)^k}{{}_a q_k! \cdot q^{(a-1)k+T(a)}}.$$

We see that

$$\frac{|{}_a C_{k+1}|}{|{}_a C_k|} = \frac{1}{|(q^{k+1} - 1) \cdot q^{a-1}|}.$$

Given that $k \geq a$ and $|q| > 1$, we let a be sufficiently large such that $|q^{k+1} - 1| > \frac{1}{2}|q|^{k+1} \geq \frac{1}{2}|q|^{a+1}$ and $\frac{2}{|q|^{2a}} < \frac{1}{2}$. In such a case,

$$\frac{|{}_a C_{k+1}|}{|{}_a C_k|} = \frac{1}{|(q^{k+1} - 1) \cdot q^{a-1}|} < \frac{2}{|q|^{a+1} \cdot |q|^{a-1}} = \frac{2}{|q|^{2a}} < \frac{1}{2}.$$

The infinite sum is bounded by a geometric series with ratio $\frac{1}{2}$:

$$\left| \sum_{k=a}^{\infty} {}_a C_k \right| \leq \sum_{k=a}^{\infty} |{}_a C_k| \leq 2|{}_a C_a|.$$

As $|{}_a C_a| = \left(|q^a - 1| \cdot |q|^{(3a^2-3a)/2} \right)^{-1} \leq 2|q|^{-(3a^2-a)/2}$, the limit is thus bounded

$$\lim_{a \rightarrow \infty} \left| \sum_{k=a}^{\infty} {}_a C_k \right| \leq \lim_{a \rightarrow \infty} 4|q|^{-(3a^2-a)/2} \rightarrow 0$$

for sufficiently large a , which guarantees the remainder converges to 0 as the terms in the preceding sum increases.

It follows that:

$$\begin{aligned} D_q &= 1 + \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{q^{k^2+T(k)}} + \frac{(-1)^k}{q^{k^2+T(k+1)}} \right) = 1 + \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{q^{(3k^2-k)/2}} + \frac{(-1)^k}{q^{(3k^2+k)/2}} \right) \\ &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{(3k^2-k)/2}}. \end{aligned}$$

□

References

- [1] Janot de Stainville. *Mélanges d'Analyse Algébrique et de Géométrie*. Veuve Courcier, 1815, pp. 340–341.
- [2] A. V. Lotosky. “Sur l’Irrationalité d’un Produit Infini”. In: *Matematicheskii Sbornik* 54 (1943), pp. 262–272.
- [3] Peter Bundschuh. “Arithmetische Untersuchungen unendlicher Produkte”. In: *Invent Math* 6 (1969), pp. 275–295. DOI: <https://doi.org/10.1007/BF01425419>.
- [4] Fabian Franklin. “Sur le développement du produit infini $(1-x)(1-x^2)(1-x^3)\dots$ ”. In: *Comptes Rendus de l’Académie des Sciences* 92 (1881), pp. 448–450.
- [5] Leonhard Euler. “Demonstratio theorematis circa ordinem in partitionibus numerorum observatum.” In: *Novi Commentarii academiae scientiarum Petropolitanae* 5 (1760), pp. 75–83.
- [6] Isaac Newton. *Arithmetica Universalis*. Benjamin Tooke, 1707, pp. 250–251.
- [7] Albert Girard. *Invention nouvelle en l’algèbre*. Guillaume Blaeuw, 1629.
- [8] G. E. Andrews. “q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra”. In: *American Math Society* (1986).